Rutgers University: Algebra Written Qualifying Exam

January 2006: Day 2 Problem 4 Solution

Exercise. Prove that every group of order 6 is either isomorphic to the cyclic group $\mathbb{Z}/6\mathbb{Z}$ or to the symmetric group S_3.

Solution.

If $|G| = 2 \cdot 3$, then by the third Sylow theorem

$$n_3 \equiv 1 \mod 3 \quad \text{and} \quad n_3 \mid 2 \quad \Longrightarrow \quad n_3 = 1$$

$$n_2 \equiv 1 \mod 2 \quad \text{and} \quad n_2 \mid 3 \quad \Longrightarrow \quad n_2 = 1 \text{ or } 3$$

If $n_3 = 1$ and $n_2 = 1$ then the Sylow 3-subgroup, P_3, and the Sylow 2-subgroup, P_2, are both normal in G.

$$P_2P_3 \subseteq G \quad \text{and} \quad |P_2P_3| = \frac{|P_2| \cdot |P_3|}{|P_2 \cap P_3|} = \frac{2 \cdot 3}{1} = |G|$$

$$\Longrightarrow \quad G = P_2P_3$$

Since $G = P_2P_3$ and both P_2, P_3 normal in G,

$$G \cong P_2 \times P_3$$

And since $|P_2| = 2$ and $|P_3| = 3$, which are both prime, P_2 and $|P_3|$ are cyclic

$$\Longrightarrow G \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$$

If $G \ncong \mathbb{Z}_6$, then $n_3 = 1$ and $n_2 = 3$.

There are $1(3-1) = 2$ elements of order 3 and $3(2-1) = 3$ elements of order 2

$$\Longrightarrow G = \{e, g_1, g_2, g_3, h, h_2\}, \quad \text{where} \quad o(g_1) = o(g_2) = o(g_3) = 2 \quad \text{and} \quad o(h) = o(h^2) = 3$$

$$h \neq g_i \quad \Longrightarrow \quad g_ih \neq \underbrace{g_i^2 = e}_{o(g_i)=2} \text{ and } hg_i \neq \underbrace{g_i^2 = e}_{o(g_i)=2} \quad \Longrightarrow \quad o(g_ih) \neq 1 \text{ and } o(hg_i) \neq 1$$

$$h \neq g_i \neq e \quad \Longrightarrow \quad h^2 \neq g_ih \neq h \text{ and } h^2 \neq hg_i \neq h \quad \Longrightarrow \quad o(g_ih) \neq 3 \text{ and } o(hg_i) \neq 3$$

$$\therefore o(g_ih) = o(hg_i) = 2 \quad \Longrightarrow g_j := g_ih = g_ih\underbrace{(g_ih^3g_i)}_{e} = \underbrace{(g_ih)^2}_{e}h^2g_i = h^2g_i \quad \text{s.t. } j \neq i$$

$$\text{and } g_k := hg_i = hg_i\underbrace{(hg_i^2h^2)}_{e} = (hg_i)^2g_ih^2 = g_ih^2 = g_jh \quad \text{s.t. } i \neq k \neq j$$

So, $h^3 = e = g_i^2$ for any i, and for distinct i, j and k:

$$g_ih = h^2g_i = g_j \qquad hg_i = g_ih^2 = g_k$$

$$g_ig_j = g_i(g_ih) = \underbrace{g_i^2}_{e}h = h \qquad g_jg_i = (h^2g_i)g_i = h^2\underbrace{g_i^2}_{e} = h^2$$

$$g_kg_i = (hg_i)g_i = h\underbrace{g_i^2}_{e} = h \qquad g_ig_k = g_i(g_ih^2) = \underbrace{g_i^2}_{e}h^2 = h^2$$

$$g_jg_k = \underbrace{(g_jg_i)}_{h^2}\underbrace{(g_ig_k)}_{h^2} = \underbrace{h^3}_{e}h = h \qquad g_kg_j = \underbrace{(g_kg_i)}_{h}\underbrace{(g_ig_j)}_{h} = h^2$$

So $(G, \cdot)$ must be defined by

$\cdot$	e	g_1	g_2	g_3	h	h^2
e	e	g_1	g_2	g_3	h	h^2
g_1	g_1	e	h^2	h	g_2	g_3
g_2	g_2	h	e	h^2	g_3	g_1
g_3	g_3	h^2	h	e	g_1	g_2
h	h	g_3	g_1	g_2	h^2	e
h^2	h^2	g_2	g_3	g_1	e	h

and $G \cong S_3$